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# SOME NUMERICAL COMPUTATIONS CONCERNING SPINOR ZETA FUNCTIONS IN GENUS 2 AT THE CENTRAL POINT

WINFRIED KOHNEN AND MICHAEL KUSS

ABSTRACT. We numerically compute the central critical values of odd quadratic character twists with respect to some small discriminants $D$ of spinor zeta functions attached to Seigel–Hecke eigenforms $F$ of genus 2 in the first few cases where $F$ does not belong to the Maass space. As a result, in the cases considered we can numerically confirm a conjecture of Böcherer, according to which these central critical values should be proportional to the squares of certain finite sums of Fourier coefficients of $F$.

## 1. INTRODUCTION

In [3], Böcherer made an interesting conjecture concerning central critical values of odd quadratic character twists of spinor zeta functions attached to cuspidal Siegel–Hecke eigenforms of genus 2.

More precisely, let $F$ be a nonzero cuspidal Hecke eigenform of even integral weight $k$ w.r.t. the Siegel modular group $\Gamma_2 := \mathrm{Sp}_2(\mathbb{Z})$ and denote by $Z_F(s)$ ($\mathrm{Re}(s) \gg 0$) its spinor zeta function. Recall [2] that $Z_F(s)$ completed with appropriate $\Gamma$-factors has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \mapsto 2k-2-s$. Let $Z_F(s,\chi_D)$ ($\mathrm{Re}(s) \gg 0$) be the twist of $Z_F(s)$ by the quadratic character $\chi_D = \left(\frac{D}{\cdot}\right)$, where $D < 0$ is a fundamental discriminant. Assume that $Z_F(s,\chi_D)$ enjoys similar analytic properties as $Z_F(s)$. Then according to [3], there should exist a constant $C_F > 0$, depending only on $F$, such that

$$Z_F(k-1,\chi_D) = C_F|D|^{1-k}\Big(\sum_{\{T>0\,|\,\mathrm{discr}\,T=D\}/\Gamma_1} \frac{a(T)}{\varepsilon(T)}\Big)^2, \tag{1}$$

where $a(T)$ ($T$ a positive definite half-integral $(2,2)$-matrix) is the $T$-th Fourier coefficient of $F$, $\varepsilon(T) := \#\{U \in \Gamma_1 \mid T[U] = T\}$ (with $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$, $T[U] = U^tTU$) is the order of the unit group of $T$ and the summation in (1) extends over all $T$ with discriminant equal to $D$, modulo the action $T \mapsto T[U]$ by $\Gamma_1$.

In [3], Böcherer proved his conjecture in the case where $F$ is the Maass lift of a Hecke eigenform $f$ of weight $2k-2$ w.r.t. $\Gamma_1$. The proof combines four inputs: i) the fact that $Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s)$, where $L(f,s)$ is the Hecke $L$-function of $f$ [5]; ii) Waldspurger's theorem [13] on the relation between central critical values of quadratic twists of $L(f,s)$ and squares of Fourier coefficients of modular forms of half-integral weight; iii) the explicit description of the Maass

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lift on the level of Fourier coefficients [2]; and finally iv) Dirichlet's classical class number formula.

Later on, Böcherer and Schulze-Pillot [4] proved an identity similar to (1) in the case of levels, where now $F$ is the Yoshida lift of an elliptic cusp form.

Also in [3], a formula like (1) in the case where $F$ is a Siegel– or Klingen–Eisenstein series was shown to be true.

The proof in all the above cases makes essential use of the fact that the spinor zeta function in question is a product of "known" $L$-series.

To the best of our knowledge, nothing regarding Böcherer's conjecture seems to be known in the case where $F$ is a "true" Siegel modular form, i.e., is not a lift of an automorphic form on $\mathrm{GL}_2$ (and so $Z_F(s)$ is not expected to split).

In the present paper, we would like to present some numerical data supporting the conjecture for small values of $D$ in the first few "nontrivial" cases when $F$ is of weight 20, 22, 24 resp. 26 and is not a Maass lift. It turns out that for those $F$ and for $D = -3, -4, -7, -8$ identity (1) *numerically* is true at least up to 5 digits with some constant $C_F > 0$ independent of $D$ (Thm., §4; numerical data are given in §5).

The first ingredient in the computation is a certain series representation (found by the first author many years ago) for central critical values of spinor zeta functions *supposing* "good" analytic properties of $Z_F(s, \chi_D)$ as required in the conjecture. We were kindly informed by D. Goldfeld that this series representation can also be derived from the more general work of Lavrik [10] when appropriately specialized. The formula for computing $Z_F(k-1, \chi_D)$ is given in §2.

Note that the holomorphic continuation of $Z_F(s, \chi_D)$ was proved in [6],[7] (using some round-about via Rankin–Dirichlet series) under the assumption that the first Fourier–Jacobi coefficient of $F$ is nonzero. The latter condition is satisfied at least for all $F$ with $k \leqslant 32$ according to Skoruppa [12]. The functional equation, however, was proved only very recently in [9].

The second main ingredient, which is entirely due to the second author, is the computation of the eigenvalues $\lambda_F(p)$ ($p$ a prime $< 1\,000$) and $\lambda_F(p^2)$ ($p$ a prime $< 71$) under the usual Hecke operators $T_p$ resp. $T_{p^2}$ of the $F$ in question, following the method of Skoruppa [12] and an appropriate C++ computer program. This is presented in §3.

In §4, the results of §§2 and 3 are combined to calculate $Z_F(k-1, \chi_D)$ for the $F$ and $D$ in question with "good" accuracy. For an estimation of the error term we use the bounds for the eigenvalues of $F$ implied by the Ramanujan–Petersson conjecture, for the latter cf. [14].

We finally remark that we have also numerically re-checked (1) using the identity given in §2 in case $F$ is of weight 20, resp. 22, and is in the Maass space. We have not included the details here.

## 2. A series representation for central values of spinor zeta functions

Let $k \in 2\mathbb{N}$ and write $S_k(\Gamma_2)$ for the space of Siegel cusp forms of weight $k$ w.r.t. $\Gamma_2$. If $F \in S_k(\Gamma_2)$ is a nonzero Hecke eigenform, we let

$$Z_F(s) = \prod_{p \text{ prime}} Z_{F,p}(p^{-s})^{-1} \qquad (\mathrm{Re}(s) \gg 0) \tag{2}$$

be the spinor zeta function of $F$, where

$$Z_{F,p}(X) = 1 - \lambda_F(p)X + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})X^2 \\ - \lambda_F(p)p^{2k-3}X^3 + p^{4k-6}X^4$$

is the local spinor polynomial at $p$ and $\lambda_F(p)$ resp. $\lambda_F(p^2)$ are the eigenvalues of $F$ under the usual Hecke operator $T_p$ resp. $T_{p^2}$.

According to Andrianov [2] the function

$$Z_F^\star(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \mapsto 2k-2-s$. It is holomorphic everywhere if $F$ is not contained in the Maass space (which is equivalent to saying $Z_F(s)$ is *not* of the form $Z_F(s) = \zeta(s-k+1)\zeta(s-k+2) \times L(f,s)$, where $f$ is a normalized cuspidal Hecke eigenform of weight $2k-2$ w.r.t. $\Gamma_1$, and $L(f,s)$ is its associated Hecke $L$-function [5]).

If $D < 0$ is a fundamental discriminant, we define the twist of $Z_F(s)$ by $\chi_D$ as

$$Z_F(s,\chi_D) := \prod_{p \text{ prime}} Z_{F,p}(\chi_D(p)p^{-s})^{-1} \qquad (\mathrm{Re}(s) \gg 0). \tag{3}$$

We denote the $n$-th coefficient of the Dirichlet series $Z_F(s,\chi_D)$ by $\lambda_{F,D}(n)$.

We put

$$Z_F^\star(s,\chi_D) := \left(\tfrac{2\pi}{|D|}\right)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s,\chi_D) \qquad (\mathrm{Re}(s) \gg 0).$$

If $F$ is in the Maass space, then by well-known properties of twists of $\zeta(s)$ and $L(f,s)$, $Z_F^\star(s,\chi_D)$ extends to an entire function, is of rapid decay for $\mathrm{Im}(s) \to \infty$ and is invariant under $s \mapsto 2k-2-s$. It is very natural to expect that the same holds for general $F$ (cf. [3]). In fact, if $F$ is not in the Maass space and the first Fourier–Jacobi coefficient of $F$ is nonzero, this was proved in [6],[7],[9] (using the fact $\|\phi_1\|^2 Z_F(s) = D_F(s)$, where $\phi_1$ is the first Fourier–Jacobi coefficient of $F$ and $D_F(s)$ is a Rankin type Dirichlet series formed out of the Fourier–Jacobi coefficients of $F$ introduced in [8]).

Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform such that $Z_F(s,\chi_D)$ has the above analytic properties. Using the integral transform

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(s-k+2)y^{-s}\mathrm{d}s = 2y^{-\frac{k}{2}+1}K_{k-2}(2\sqrt{y}) \qquad (y>0,\ c>k-2),$$

where $K_{k-2}(y)$ denotes the modified Bessel function of order $k-2$, we have for $y > 0$ and $c \gg 0$

$$\begin{aligned} &\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z_F^\star(s,\chi_D)y^{-s}\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\tfrac{2\pi}{|D|}\right)^{-2s}\Gamma(s)\Gamma(s-k+2)\sum_{n=1}^{\infty}\lambda_{F,D}(n)n^{-s}y^{-s}\mathrm{d}s \\ &= \sum_{n=1}^{\infty}\lambda_{F,D}(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(s-k+2)\left(\tfrac{4\pi^2 ny}{D^2}\right)^{-s}\mathrm{d}s \\ &= y^{-\frac{k}{2}+1}f_{F,D}(y), \end{aligned} \tag{4}$$

where

$$f_{F,D}(y) = 2\left(\tfrac{4\pi^2}{D^2}\right)^{-\frac{k}{2}+1} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-\frac{k}{2}+1} K_{k-2}\left(\tfrac{4\pi\sqrt{ny}}{|D|}\right).$$

Since $Z_F^\star(s,\chi_D)$ is holomorphic and of rapid decay for $\operatorname{Im} s \to \infty$, we may shift the path of integration in (4) to the line $c = k-1$. We replace $y$ by $\frac{1}{y}$ and apply the functional equation of $Z_F^\star(s,\chi_D)$ to obtain

$$\begin{aligned} &y^{\frac{k}{2}-1} f_{F,D}\left(\tfrac{1}{y}\right) \\ &= \tfrac{1}{2\pi i} \int_{k-1-i\infty}^{k-1+i\infty} Z_F^\star(s,\chi_D) y^s \mathrm{d}s = \tfrac{1}{2\pi i} \int_{k-1-i\infty}^{k-1+i\infty} Z_F^\star(2k-2-s,\chi_D) y^s \mathrm{d}s \\ &= \tfrac{1}{2\pi i} \int_{k-1-i\infty}^{k-1+i\infty} Z_F^\star(s,\chi_D) y^{2k-2-s} \mathrm{d}s = y^{\frac{3}{2}k-1} f_{F,D}(y), \end{aligned}$$

i.e., the function $f_{F,D}(y)$ satisfies the functional equation $f_{F,D}(\frac{1}{y}) = y^k f_{F,D}(y)$.

Using the usual splitting trick and the formula

$$2\int_0^\infty K_{k-2}(2\sqrt{y}) y^{s-\frac{k}{2}} \mathrm{d}y = \Gamma(s)\Gamma(s-k+2) \qquad (\operatorname{Re}(s) > k-2),$$

we conclude for $\operatorname{Re}(s) \gg 0$ that

(5)
$$\begin{aligned} Z_F^\star(s,\chi_D) &= 2\left(\tfrac{2\pi}{|D|}\right)^{-2s} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-s} \int_0^\infty K_{k-2}(2\sqrt{y}) y^{s-\frac{k}{2}} \mathrm{d}y \\ &= 2\left(\tfrac{2\pi}{|D|}\right)^{-2s} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-s} \left(\tfrac{4\pi^2 n}{D^2}\right)^{s-\frac{k}{2}+1} \int_0^\infty K_{k-2}\left(\tfrac{4\pi\sqrt{ny}}{|D|}\right) y^{s-\frac{k}{2}} \mathrm{d}y \\ &= \int_0^\infty f_{F,D}(y) y^{s-\frac{k}{2}} \mathrm{d}y = \int_1^\infty f_{F,D}(y)\left(y^{\frac{3}{2}k-2-s} + y^{s-\frac{k}{2}}\right) \mathrm{d}y. \end{aligned}$$

As $f_{F,D}(y)$ is of exponential decay for $y \to \infty$, the right hand side of (5) has a holomorphic continuation to the whole complex plane, and (5) is valid for all $s \in \mathbb{C}$.

Setting $s = k-1$ in (5), we get the formulas

(6)
$$\begin{aligned} Z_F^\star(k-1,\chi_D) &= 4\left(\tfrac{4\pi^2}{D^2}\right)^{-\frac{k}{2}+1} \int_1^\infty \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-\frac{k}{2}+1} K_{k-2}\left(\tfrac{4\pi\sqrt{ny}}{|D|}\right) y^{\frac{k}{2}-1} \mathrm{d}y \\ &= 4\left(\tfrac{4\pi^2}{D^2}\right)^{-\frac{k}{2}+1} \sum_{n=1}^{\infty} \int_1^\infty \lambda_{F,D}(n) n^{-\frac{k}{2}+1} K_{k-2}\left(\tfrac{4\pi\sqrt{ny}}{|D|}\right) y^{\frac{k}{2}-1} \mathrm{d}y \\ &= 4\left(\tfrac{4\pi^2}{D^2}\right)^{-\frac{k}{2}+1} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-k+1} \int_n^\infty K_{k-2}\left(\tfrac{4\pi\sqrt{y}}{|D|}\right) y^{\frac{k}{2}-1} \mathrm{d}y. \end{aligned}$$

Hence

(7)
$$Z_F(k-1,\chi_D) = \tfrac{4(2\pi)^k}{|D|^k (k-2)!} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-k+1} \int_n^\infty K_{k-2}\left(\tfrac{4\pi\sqrt{y}}{|D|}\right) y^{\frac{k}{2}-1} \mathrm{d}y,$$

where the exponential decay of $K_{k-2}(y)$ for $y \to \infty$ justifies the interchange of summation and integration in (6).

## 3. Numerical computations

Let $M_k(\Gamma_1)$ be the space of elliptic modular forms of weight $k$ w.r.t. $\Gamma_1$ and $S_k(\Gamma_1)$ be the subspace of cusp forms in $M_k(\Gamma_1)$. For $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$, write $q = \exp(2\pi i\tau)$, and let

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

be the Ramanujan $\Delta$-function in $S_{12}(\Gamma_1)$ and

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \qquad (k \in \mathbb{Z},\ k \geqslant 2,\ \sigma_{2k-1}(n) = \textstyle\sum_{d|n} d^{2k-1},$$
$$B_{2k} = 2k\text{th Bernoulli number})$$

be the normalized Eisenstein series in $M_{2k}(\Gamma_1)$.

If $J_{k,1}^{\mathrm{cusp}}$ denotes the space of Jacobi cusp forms on $\Gamma_1$ of index 1 and weight $k$, the Maass space [11] is the image of the Hecke equivariant embedding $V : J_{k,1}^{\mathrm{cusp}} \hookrightarrow S_k(\Gamma_2)$ defined by

$$\begin{aligned} \phi = & \sum_{\substack{D,r \in \mathbb{Z}, D<0 \\ D \equiv r^2 \bmod 4}} C_\phi(D) q^{(r^2-D)/4} \zeta^r \\ \longmapsto & \sum_{\substack{n,r,m \in \mathbb{Z} \\ r^2-4mn<0, \\ n,m>0}} a(n,r,m) q^n \zeta^r q'^m, \end{aligned}$$

where

$$a(n,r,m) := \sum_{d|(n,r,m)} d^{k-1} C_\phi\Big(\frac{r^2-4mn}{d^2}\Big)$$

and

$$\zeta = \exp(2\pi i z)(z \in \mathbb{C}), \qquad q' = \exp(2\pi i \tau')(\tau' \in \mathbb{C}, \mathrm{Im}(\tau') > 0).$$

By $\phi_{10}$ resp. $\phi_{12}$ we denote the Jacobi cusp forms in the one-dimensional spaces $J_{10,1}^{\mathrm{cusp}}$, resp. $J_{12,1}^{\mathrm{cusp}}$, normalized to $C(-3) = 1$.

The first cuspidal Hecke eigenforms for genus 2 that do not belong to the Maass space appear in weight 20, 22, 24, resp. 26, and are denoted $\Upsilon_{20}, \ldots, \Upsilon_{26b}$ in [12]. In [12], Skoruppa gives explicit formulas for them (involving the forms $V(\phi)$, where $\phi$ are appropriate Jacobi forms) and calculates some of their Fourier coefficients. Note that there is a misprint in the formula for $\Upsilon_{22}$; the corrected formula is

$$\begin{aligned} \Upsilon_{22} = & -2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 1423 \cdot V(\phi_{10}) V(\phi_{12}) \\ & + V\big(-\tfrac{5}{2\cdot 3}\phi_{12} E_{10} + \tfrac{11}{2\cdot 3}\phi_{10} E_6^2 + 2^4 \cdot 3 \cdot 61 \cdot \phi_{10}\Delta\big). \end{aligned}$$

To compute the coefficients of the relevant Jacobi forms $\phi$, we proceed slightly differently from [12] and try to avoid multiplication of Jacobi modular forms with elliptic modular forms. More precisely, the operator $\mathcal{D}_{2\nu}$ is defined by

$$\mathcal{D}_{2\nu}\phi := \sum_{n=0}^{\infty}\Big(\sum_{r} p_{2\nu}^{(k-1)}(r,nm)c(n,r)\Big)q^n \qquad (\nu\in\mathbb{Z},\ \nu\geqslant 0)$$

where $\phi=\sum_{n,r}c(n,r)q^n\zeta^r\in J_{k,m}^{\mathrm{cusp}}$ and

$$\frac{(k-\nu-2)!}{(2\nu)!(k-2)!}\,p_{2\nu}^{(k-1)}(r,n) = \text{coefficient of } t^{2\nu} \text{ in } (1-rt+nt^2)^{-k+1}$$

maps $J_{k,1}^{\mathrm{cusp}}$ to $S_{k+2\nu}(\Gamma_1)$ [5].

We consider the system of equations $\{\mathcal{D}_{2\nu}(f)=g_{f,\nu}\}$ where $f$ is one of the Jacobi forms

$$\phi_{10},\phi_{10}E_4,\phi_{10}E_6,\phi_{10}E_{10},\phi_{10}E_{14},\phi_{10}E_{16},\phi_{10}\Delta,\phi_{10}E_6^2,\phi_{10}E_8^2,\phi_{10}\Delta E_4,$$
$$\phi_{12},\phi_{12}E_8,\phi_{12}E_{10},\phi_{12}E_6^2,\phi_{12}\Delta,\phi_{12}E_{14},$$

$\nu\in\{0,2,4\}$ and $g_{f,\nu}$ is the corresponding elliptic modular form which is determined by its first coefficients, e.g., we have

$$\mathcal{D}_0(\phi_{10})=0,\qquad \mathcal{D}_2(\phi_{10})=20\Delta,\qquad \mathcal{D}_4(\phi_{10})=0,$$
$$\mathcal{D}_0(\phi_{12})=12\Delta,\qquad \mathcal{D}_2(\phi_{12})=0,\qquad \mathcal{D}_4(\phi_{12})=196\Delta E_4.$$

We solve the system recursively for the Fourier coefficients of the Jacobi forms. (To start the recursion the first Fourier coefficients of $\phi_{10}$, $\phi_{12}$ are taken from [5].) This method needs only $O(|D|^{\frac{3}{2}})$ operations to calculate a complete table of Fourier coefficients up to a "large" discriminant. Hence it is less "expensive" than the usual multiplication of Jacobi forms and elliptic modular forms $(O(|D|^2))$.

Proceeding in this way and using a C++ computer program, we computed the Fourier coefficients $C(D)$ of the Jacobi forms in question for $|D|\leqslant 3\,000\,000$. Then we are able to compute any Fourier coefficient $a(n,r,m)$ of $\Upsilon_{20},\ \dots,\ \Upsilon_{26b}$ with discriminant $4mn-r^2\leqslant 3\,000\,000$.

In [12] Skoruppa calculates the eigenvalues $\lambda_F(p)$, $\lambda_F(p^2)$ ($p$ prime) of a Hecke eigenform

$$F=\sum_{\substack{r,n,m\in\mathbb{Z},\\ r^2-4mn<0,\\ n,m>0}} a(n,r,m)q^n\zeta^r q'^m \in S_k(\Gamma_2)$$

by means of the formulas

$$\lambda_F(p)a(1,1,1)=a(p,p,p)+p^{k-2}\big(1+\big(\tfrac{p}{3}\big)\big)a(1,1,1)$$

and

$$\begin{aligned}\lambda_F(p^2)&a(1,1,1)\\ &=\Big[\lambda_F(p)^2-\lambda_F(p)p^{k-2}\big(1+\big(\tfrac{p}{3}\big)\big)-p^{2k-3}+p^{2k-4}\big(\big(\tfrac{p}{3}\big)+\big(\tfrac{p}{3}\big)^2\big)\Big]a(1,1,1)\\ &\quad -p^{k-2}a(1,p,p^2)-p^{k-2}\sum_{\substack{\nu \bmod p,\\ 1+\nu+\nu^2\not\equiv 0 \bmod p}} a(1+\nu+\nu^2,p(1+2\nu),p^2),\end{aligned}$$

which are based on Andrianov's results in [2].

Using another C++ computer program, we computed the eigenvalues $\lambda_F(p)$ for $p < 1\,000$ prime and $\lambda_F(p^2)$ for $p < 71$ prime of $F = \Upsilon_{20}, \ldots, \Upsilon_{26b}$ from the above formulas.

## 4. Summing up

By (7) we have

$$Z_F(k-1,\chi_D) = \sum_{n=1}^{\infty} \lambda_{F,D}(n) g_D(n),$$

where

$$g_D(n) = \tfrac{4(2\pi)^k}{|D|^k (k-2)!}\, n^{-k+1} \int_n^{\infty} K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y. \tag{8}$$

Now $g_D(n)$ is of exponential decay for $n \to \infty$ and $\lambda_{F,D}(n)$ is of polynomial growth. Thus for a numerical approximation of $Z_F(k-1,\chi_D)$ it is important to calculate as many terms as possible in the sum for *small* $n$ (say $n \leqslant N$ for some $N$ — we will later choose $N = 4000$), while for *large* $n$ ($n > N$) the total sum of *all* terms with $n > N$ is rather small. Hence we approximate $Z_F(k-1,\chi_D)$ by

$$\mathcal{Z}_{F,D}(k-1) = \sum_{\substack{1 \leqslant n \leqslant N \\ n \text{ has no prime} \\ \text{divisor} > 1\,000}} \lambda_{F,D}(n) g_D(n),$$

where the values of $\lambda_{F,D}(n)$ can be calculated from the Euler product of $Z_{F,D}(s,\chi_D)$ for $n < 71^2$.

Suppose there are positive constants $C_1, C_2, \alpha, \beta$ such that the estimates $|\lambda_F(p)| \leqslant C_1 \cdot p^{\alpha}$ ($p$ prime) and $|\lambda_F(n)| \leqslant C_2 \cdot n^{\beta}$ ($n > N$) hold. Then the error term

$$\varepsilon(F,D) = Z_{F,D}(k-1) - \mathcal{Z}_{F,D}(k-1)$$

can be estimated by

$$\begin{aligned} |\varepsilon(F,D)| \leqslant &\sum_{\substack{p>1\,000 \\ p \text{ prime} \\ 1 \leqslant \nu \leqslant N/p}} |\lambda_{F,D}(\nu p)| g_D(\nu p) \\ &+ \sum_{n>N} |\lambda_{F,D}(\nu n)| \int_n^{\infty} K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y. \end{aligned} \tag{9}$$

Suppose now that $N < 1007^2$. Then clearly for the first sum $\sum_1$ in the above equation we have the estimate

$$\begin{aligned} \sum\nolimits_1 &= \sum_{\substack{p>1\,000 \\ p \text{ prime} \\ 1 \leqslant \nu \leqslant N/p}} \big|\lambda_{F,D}(\nu)\lambda_F(p)\big|\, g_D(\nu p) \\ &\leqslant C_1 \sum_{\substack{p>1\,000 \\ p \text{ prime} \\ 1 \leqslant \nu \leqslant N/p}} |\lambda_{F,D}(\nu)| p^{\alpha} g_D(\nu p). \end{aligned}$$

The second sum $\sum_2$ in (9) satisfies

$$\begin{aligned}\tfrac{|D|^k(k-2)!}{4(2\pi)^k}\sum\nolimits_2 &= \sum_{n>N} |\lambda_{F,D}(n)| n^{-k+1} \int_n^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1}\mathrm{d}y \\ &\leqslant C_2 \sum_{n>N} n^{\beta-k+1} \int_n^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1}\mathrm{d}y \\ &\leqslant C_2 \sum_{n>N} \int_n^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\beta-\frac{k}{2}}\mathrm{d}y \\ &\leqslant C_2 \sum_{n>N}\sum_{m\geqslant n} \int_m^{m+1} K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\beta-\frac{k}{2}}\mathrm{d}y \\ &\leqslant C_2 \sum_{m>N} (m-N) \int_m^{m+1} K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\beta-\frac{k}{2}}\mathrm{d}y \\ &\leqslant C_2 \int_{N+1}^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\beta-\frac{k}{2}}(y-N)\mathrm{d}y.\end{aligned}$$

For the estimation of the dominating term $\sum_1$ in $\varepsilon(F,D)$ we use the result of Weissauer [14] that any eigenform $F \in S_k(\Gamma_2)$ which does not belong to the Maass space fulfills the Ramanujan–Petersson conjecture (i.e., all complex roots of $Z_{F,p}$ have absolute value $p^{\frac{3}{2}-k}$). Thus we have to choose $C_1 = 4$, $\alpha = k - \frac{3}{2}$ to obtain the *best* estimate for $\sum_1$ possible by our methods.

The contribution of $\sum_2$ to $\varepsilon(F,D)$ is absorbed by $\sum_1$ if $N$ is large enough, so we do not have to use the optimal estimate for $\lambda_F(n)$. One obtains a very crude (but simple and for our purpose sufficient) estimate for $\lambda_F(n)$ from the Ramanujan–Petersson conjecture if one uses $\sigma_0(n) \leqslant n$, namely

$$|\lambda_F(n)| \leqslant \sum_{d|n} \sigma_0(d)\sigma_0(\tfrac{n}{d}) n^{k-\frac{3}{2}} \leqslant \sum_{d|n} n^{k-\frac{1}{2}} = \sigma_0(n) n^{k-\frac{1}{2}} \leqslant n^{k+\frac{1}{2}}.$$

Thus we set $C_2 = 1$ and $\beta = k + \frac{1}{2}$.

We choose $N = 4\,000$ (then $\sum_2$ is dominated by $\sum_1$ for the $D$ in question) and calculate the numerical approximations of $Z_F(k-1, \chi_D)$ and the corresponding error terms using Mathematica. From (1) we computed the constants $C_F$ for $F = \Upsilon_{20}$, ..., $\Upsilon_{26b}$ and $D = -3, -4, -7, -8$. The numerical results have been checked using Maple.

We obtain

**Theorem.** *For* $F = \Upsilon_{20}, \ldots, \Upsilon_{26b}$ *there are constants* $C_F$ *such that equation* (1) *(i.e., Böcherer's conjecture) holds for* $D = -3, -, 4 - 7, -8$ *numerically up to* 5 *digits.*

## 5. Numerical data

TABLE 1. Approximate constants $C_F$ for $F = \Upsilon_{20}, \Upsilon_{22}$

| $D$ | $C_{\Upsilon_{20}}$ | $C_{\Upsilon_{22}}$ |
|---|---|---|
| $-3$ | $2.067215202868\,8 \cdot 10^{11} \pm 0.5 \cdot 10^{-2}$ | $1.305668526829\,0 \cdot 10^{12} \pm 0.5 \cdot 10^{-1}$ |
| $-4$ | $2.067215202868\,8 \cdot 10^{11} \pm 0.5 \cdot 10^{-2}$ | $1.305668526829\,0 \cdot 10^{12} \pm 0.5 \cdot 10^{-1}$ |
| $-7$ | $2.067215202\,9206 \cdot 10^{11} \pm 2.9 \cdot 10^{1}$ | $1.3056685268\,295 \cdot 10^{12} \pm 1.1 \cdot 10^{1}$ |
| $-8$ | $2.067215202\,8644 \cdot 10^{11} \pm 3.1 \cdot 10^{1}$ | $1.305668\,5179067 \cdot 10^{12} \pm 1.1 \cdot 10^{6}$ |

TABLE 2. Approximate constants $C_F$ for $F = \Upsilon_{24a}, \Upsilon_{24b}$

| $D$ | $C_{\Upsilon_{24a}}$ | $C_{\Upsilon_{24b}}$ |
|---|---|---|
| $-3$ | $1.095337244519\,4 \cdot 10^{13} \pm 0.5 \cdot 10^{0}$ | $6.138805283929\,6 \cdot 10^{11} \pm 0.5 \cdot 10^{-2}$ |
| $-4$ | $1.095337244519\,4 \cdot 10^{13} \pm 0.5 \cdot 10^{0}$ | $6.138805283929\,6 \cdot 10^{11} \pm 0.5 \cdot 10^{-2}$ |
| $-7$ | $1.0953372445\,111 \cdot 10^{13} \pm 8.7 \cdot 10^{2}$ | $6.1388052\,891963 \cdot 10^{11} \pm 9.3 \cdot 10^{3}$ |
| $-8$ | $1.09533724\,45386 \cdot 10^{13} \pm 2.6 \cdot 10^{4}$ | $6.1388\,034612038 \cdot 10^{11} \pm 1.3 \cdot 10^{6}$ |

TABLE 3. Approximate constants $C_F$ for $F = \Upsilon_{26a}, \Upsilon_{26b}$

| $D$ | $C_{\Upsilon_{26a}}$ | $C_{\Upsilon_{26b}}$ |
|---|---|---|
| $-3$ | $9.615528574589\,1 \cdot 10^{13} \pm 0.5 \cdot 10^{0}$ | $6.232883950541\,7 \cdot 10^{12} \pm 0.5 \cdot 10^{-1}$ |
| $-4$ | $9.615528574589\,1 \cdot 10^{13} \pm 0.5 \cdot 10^{0}$ | $6.232883950541\,7 \cdot 10^{12} \pm 0.5 \cdot 10^{-1}$ |
| $-7$ | $9.61552857\,46522 \cdot 10^{13} \pm 1.1 \cdot 10^{4}$ | $6.232883950\,5729 \cdot 10^{12} \pm 2.3 \cdot 10^{2}$ |
| $-8$ | $9.615528\,5333968 \cdot 10^{13} \pm 8.2 \cdot 10^{6}$ | $6.232883\,9821394 \cdot 10^{12} \pm 1.6 \cdot 10^{5}$ |

TABLE 4. The first Fourier coefficients of $\Upsilon_{20}, \dots, \Upsilon_{26b}$

| $D$ | $n, r, m$ | $\Upsilon_{20}$ | $\Upsilon_{22}$ | $\Upsilon_{24a}$ | $\Upsilon_{24b}$ | $\Upsilon_{26a}$ | $\Upsilon_{26b}$ |
|---|---|---|---|---|---|---|---|
| −3 | 1, 1, 1 | 1 | 1 | 1 | 3 | 1 | 3 |
| −4 | 1, 0, 1 | 4 | −12 | −16 | 76 | −8 | 124 |
| −7 | 1, 1, 2 | 56 | 1 344 | 4 408 | −616 | −7 456 | 51 632 |
| −8 | 1, 0, 2 | 2 616 | 216 | 44 256 | −2 904 | 15 216 | −109 752 |
| −11 | 1, 1, 3 | −55 077 | 409 779 | −1 147 701 | 2 122 593 | −1 180 509 | 7 299 177 |
| −12 | 1, 0, 3 | 408 832 | 468 448 | −378 272 | 11 995 968 | 3 505 408 | −39 833 376 |
| −12 | 2, 2, 2 | −840 960 | −2 215 680 | −795 324 | 18 309 504 | 9 218 340 | 495 227 520 |

TABLE 5. The first eigenvalues of $\Upsilon_{20}$

| $n$ | $\lambda(n)$ |
|---|---|
| 2 | −840 960 |
| 3 | 346 935 960 |
| 5 | −5 232 247 240 500 |
| 7 | 2 617 414 076 964 400 |
| 11 | 1 427 823 701 421 564 744 |
| 13 | −83 773 835 478 688 698 980 |
| 17 | 14 156 088 476 175 218 899 620 |
| 19 | 146 957 560 176 221 097 673 720 |
| 23 | −7 159 245 922 546 757 692 913 520 |
| 29 | 1 055 528 218 470 800 414 110 149 180 |
| 31 | 4 031 470 549 468 367 403 585 068 224 |
| 37 | −154 882 657 977 740 251 483 442 365 940 |
| 41 | 1 126 683 124 934 949 617 518 831 346 964 |
| 43 | 74 572 686 686 194 644 813 168 430 600 |
| 47 | −13 773 335 595 379 978 013 820 602 730 720 |
| 53 | 29 292 488 702 536 161 643 591 933 657 260 |
| 59 | 521 943 213 201 995 351 655 113 144 025 960 |
| 61 | 896 978 197 899 858 751 399 574 623 768 444 |
| 67 | −2 921 787 486 641 381 474 027 809 454 434 280 |
| $2^2$ | 248 256 200 704 |
| $3^2$ | −452 051 040 393 665 991 |
| $5^2$ | −94 655 785 156 653 029 446 859 375 |
| $7^2$ | −5 501 629 950 184 780 949 434 983 315 951 |
| $11^2$ | −126 258 221 861 417 704 499 584 077 355 164 268 151 |
| $13^2$ | 2 528 254 555 352 510 520 887 488 261 241 887 242 369 |
| $17^2$ | 262 144 933 510 286 336 089 464 293 262 250 165 947 750 889 |
| $19^2$ | −283 417 759 450 334 375 466 210 009 895 464 677 379 295 086 759 |
| $23^2$ | 127 862 428 522 278 879 932 688 110 084 314 434 400 497 569 566 129 |
| $29^2$ | 408 550 299 154 535 330 723 926 336 201 059 419 422 405 306 949 883 361 |
| $31^2$ | −9 417 686 481 892 622 568 784 061 821 415 683 057 728 289 096 885 473 471 |
| $37^2$ | 4 270 657 975 661 931 417 960 508 434 757 260 969 748 219 593 839 247 065 169 |
| $41^2$ | 129 620 395 091 878 626 890 240 343 719 327 738 119 688 391 311 944 613 269 369 |
| $43^2$ | −2 118 391 905 744 174 698 890 014 439 813 915 105 652 042 393 393 982 400 772 151 |
| $47^2$ | 10 717 867 956 150 312 430 187 083 192 735 560 357 439 349 298 395 760 667 696 609 |
| $53^2$ | −6 359 983 052 359 692 969 866 068 986 893 310 598 482 880 773 029 488 944 413 754 191 |
| $59^2$ | 159 291 906 542 794 821 742 879 348 124 552 646 753 906 149 121 778 952 350 318 431 721 |
| $61^2$ | −653 805 853 261 332 407 170 328 486 766 159 640 869 797 840 457 778 124 369 821 593 951 |
| $67^2$ | 25 254 882 862 606 589 034 647 035 623 760 404 781 292 970 925 413 106 240 956 567 868 089 |

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UNIVERSITÄT HEIDELBERG, MATHEMATISCHES INSTITUT, IM NEUENHEIMER FELD 288, D-69120 HEIDELBERG, GERMANY

*E-mail address*: winfried@mathi.uni-heidelberg.de

UNIVERSITÄT HEIDELBERG, MATHEMATISCHES INSTITUT, IM NEUENHEIMER FELD 288, D-69120 HEIDELBERG, GERMANY

*E-mail address*: michael.kuss@urz.uni-heidelberg.de